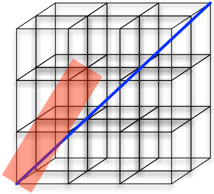
The Fat Strut Problem

V. A. Vaishampayan

(joint work with N. J. A. Sloane and S. Costa)

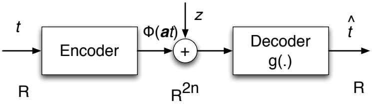
March 19, 2010

The Fat Strut Problem



- Given $\mathbf{a} \in \mathbb{Z}^n$, $\texttt{Strut}(\mathbf{a})$ is a cylinder with axis $\mathcal{L}_{\mathbf{a}} := \{t\mathbf{a},\ t \in \mathbb{R}\}$ which contains no $\mathbb{Z}^n$ points.
- $\texttt{FatStrut}(\mathbf{a})$: $\texttt{Strut}(\mathbf{a})$ with largest radius.
- Find $\mathbf{a}$ such that radius of $\texttt{FatStrut}(\mathbf{a})$ is maximized.
- $d(\mathbf{n}, \mathcal{L}_{\mathbf{a}}) := \min_{t \in \mathbb{R}} \|\mathbf{a}t - \mathbf{n}\|$: shortest distance between $\mathcal{L}_{\mathbf{a}}$ and $\mathbf{n} \in \mathbb{Z}^n$.
- Equivalent problem statement: Find $\mathbf{a}$ such that $\min_{\mathbf{n} \neq k\mathbf{a}, k \in \mathbb{Z}} d(\mathbf{n}, \mathcal{L}_{\mathbf{a}})$ is maximized.

Motivation: Combined Source and Channel Coding



- Message t to be sent over Gaussian vector channel subject to power constraint.
- Encoder $\mathbb{R} \to \mathbb{R}^{2n}$, $t \to \Phi(2\pi \mathbf{a} t/\sqrt{n})$, $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$
- $\Phi \ : \ \mathbb{R}^n \to \mathbb{R}^{2n}$
  $\Phi(\mathbf{x}) = \frac{1}{\sqrt{n}} \left(\cos(\sqrt{n} x_1), \sin(\sqrt{n} x_1), \ldots, \cos(\sqrt{n} x_n), \sin(\sqrt{n} x_n) \right)$.
- Channel output:
  $$\mathbf{y} = \Phi(\mathbf{a}t) + \mathbf{z}$$
  where $\mathbf{z}$ independent Gaussian $N(0, \sigma_z^2)$.
- Decoder: $\hat{t} = g(\mathbf{y})$. Minimize MSE $E[(t - \hat{t})^2]$ subject to $E(\Phi(\mathbf{a}t)^{tr} \Phi(\mathbf{a}t)) \leqslant P$.

Geometric Parameters of Interest...

- The signal set, a helix, is the image of

$$\{\mathbf{a}t + \mathbf{n}, \mathbf{n} \in \mathbb{Z}^k, \ t \in \mathbb{R}\}$$

  under Φ and is geometrically uniform.
- Stretch: $\|d\Phi/dt\| = (2\pi/\sqrt{n})\|\mathbf{a}\|$.
- Minimum distance between laps: ρ_{min}
- $\frac{4}{\pi^2}\epsilon_{\mathbf{a}} \leqslant \rho_{min} \leqslant \epsilon_{\mathbf{a}}$ where...
- $\epsilon_{\mathbf{a}} = \min_{\mathbf{n} \neq k\mathbf{a}, k \in \mathbb{Z}} \min_t \|\mathbf{a}t - \mathbf{n}\|$. Fat Strut Radius

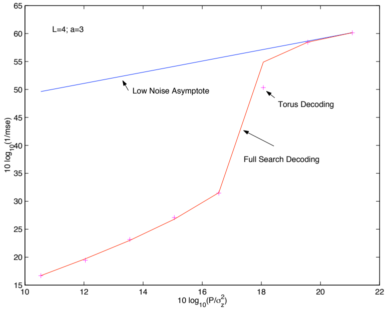
Analysis of MSE...

- $E((t-\hat{t})^2 | \text{Small Noise Event}) \approx C/\|\mathbf{a}\|^2$.
- $Pr(\text{Small Noise Event}) \uparrow$ as $\epsilon_{\mathbf{a}} \uparrow$.

Problem Def

Design Goal: Maximize $\|\mathbf{a}\|^2$ subject to lower bound on $\epsilon_{\mathbf{a}}$.

Design Criteria

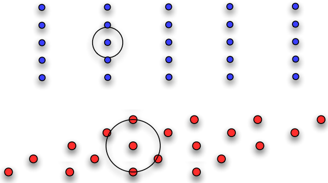


- Large $\|\mathbf{a}\|^2$ means better "small noise" performance.
- Large $\epsilon_{\mathbf{a}}$ implies sharp rolloff occurs at smaller snr.

Outline

- An upper bound.
- Non-constructive lower bound.
- Construction.

Fat Strut Radius and Projection Lattice $\Lambda_{\mathbf{a}}$



- $\epsilon_{\mathbf{a}} = d_{min}(\Lambda_{\mathbf{a}})$ where $\Lambda_{\mathbf{a}} = \mathcal{P}_{\mathbf{a}}\mathbb{Z}^n$. $\mathcal{P}_{\mathbf{a}} = \mathbf{I} - \mathbf{a}\mathbf{a}^t/\|\mathbf{a}\|^2$. Projection
- Volume of fundamental region of $\Lambda_{\mathbf{a}}$: $1/\|\mathbf{a}\|$.
- V_n: volume of n-dim sphere of unit radius.
- Packing density: $\Delta(\Lambda_{\mathbf{a}}) = V_{n-1}(\epsilon_{\mathbf{a}}/2)^{n-1}/(1/\|\mathbf{a}\|) \leqslant \Delta^*_{n-1}$.
- $\epsilon(\mathbf{a})^{n-1} \leqslant \frac{\Delta^*_{n-1}2^{n-1}}{V_{n-1}\|\mathbf{a}\|}$.
- Hence choose $\mathbf{a}$ to maximize $\Delta(\Lambda_{\mathbf{a}})$.

at&t

Achievable Lower Bound Prelim: Constraints and Symmetries

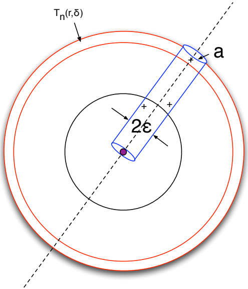
- $\mathbf{a}$ must be primitive, else $\texttt{RadiusFatStrut}(\mathbf{a}) = 0$.

  $gcd(a_1, a_2, \ldots, a_n) = 1$.
- $d(\mathbf{x}, \mathcal{L}_{\mathbf{a}}) = d(\mathbf{a} - \mathbf{x}, \mathcal{L}_{\mathbf{a}})$.

  $\mathcal{L}_{\mathbf{a}}$: line through 0 and $\mathbf{a}$.

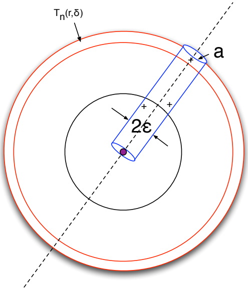
  $d(\mathbf{x}, \mathcal{L}_{\mathbf{a}})$ is shortest distance between point $\mathbf{x}$ and line $\mathcal{L}_{\mathbf{a}}$.
- For $\mathbf{x} \neq 0$ if $d(\mathbf{x}, \mathcal{L}_{\mathbf{a}})$ is minimum then $\mathbf{x}$ is primitive.

Achievable Bound: Notation and Proof Outline



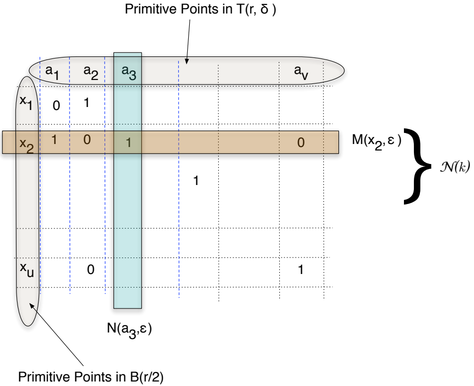
- $\mathbb{Z}^{n\#}$: Primitive vectors in $\mathbb{Z}^n$.
- $N(\mathbf{a}, \epsilon) = |\{\mathbf{x} \in \mathbb{Z}^{n\#} : \ \|\mathbf{x}\| \leqslant \|\mathbf{a}\|/2, \ d(\mathbf{x}, \mathcal{L}_{\mathbf{a}}) < \epsilon\}|$.
- $\bar{N}(\epsilon, r, \delta) = \text{Average}(\ N(\mathbf{a}, \epsilon) \ )$
- If $\bar{N}(\epsilon, r, \delta) < 1$ a strut exists with length $\approx r$ and radius ϵ.

Achievable Bound: Notation and Remark



- $T_n(r, \delta)$: Shell of thickness $r\delta$ and inner radius r.
- $\bar{N}(\epsilon, r, \delta)$ is difficult to estimate.

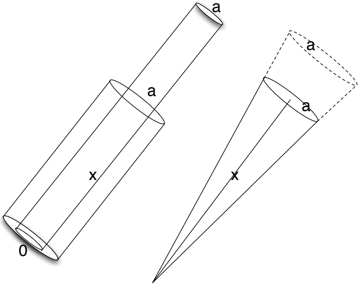
Estimating $\bar{N}(\epsilon, r, \delta)$



Column sums are difficult to estimate analytically.

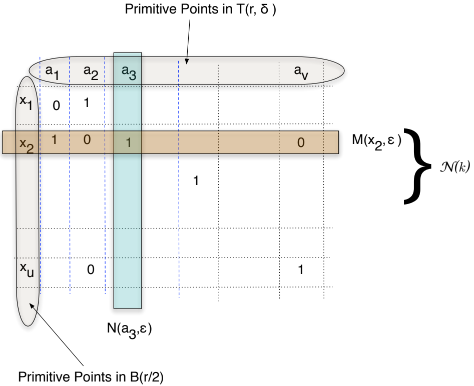
Row sums can be estimated analytically.

Why estimate column sums rather than row sums?



- Columns: Cylinders of radius ϵ
- For given ϵ, $vol \sim \|\mathbf{a}\|$.
- Row: Cone $\bigcap$ shell.
- For given ϵ, $vol \sim \|\mathbf{a}\|^n$.

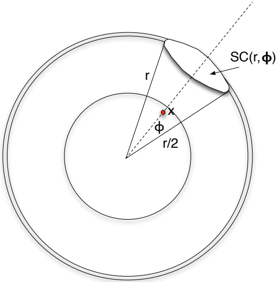
Estimating $\bar{N}(\epsilon, r, \delta)$



- $\mathcal{N}$: Total Number of 1's in above table.
- $\mathcal{N}(k)$: Contribution from $\|\mathbf{x}\|^2 = k$. $\mathcal{N} = \sum_k \mathcal{N}(k)$
- $M(\mathbf{x}, \epsilon)$ numbers of 1's in row $\mathbf{x}$. $\mathcal{N}(k) = \sum_{\mathbf{x}:\|\mathbf{x}\|=k} M(\mathbf{x}, \epsilon)$.

at&t

Estimating $\mathcal{N}(k)$



- $\sin(\phi) = \epsilon/\sqrt{k}$
- $\mathcal{N}(k) \leqslant (\Theta_n(k)) \left[Area(SC(r, \epsilon/\sqrt{k})\right] r\delta/\zeta(n)$ $\Theta_n(k)$: No. of $\mathbb{Z}^n$ points in kth shell.
- $Area(SC(r, \epsilon/\sqrt{k})) \approx V_{n-1}(\epsilon/\sqrt{k})^{n-1} r^{n-1}$ for small ϵ. V_n: vol, unit sphere in $\mathbb{R}^n$
- $\Theta_n(k) \approx V_n k^{n/2}$

Thus

$$\bar{N}(\epsilon, r, \delta) \leqslant V_{n-1} \frac{\epsilon^{n-1} r}{2\sqrt{1-\epsilon^2}} (1 + o_r(1)).$$

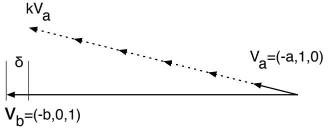
- Packing density

$$\Delta_{n-1} \geqslant (1-\gamma) \frac{1}{2^{n-2}}$$

  is achievable for any $\gamma > 0$ and r suitably large.
- Compare to Minkowski-Hlawka achievability theorem:

$$\Delta_{n-1} \approx \frac{\zeta(n-1)}{2^{n-2}}$$

Construction: $\mathbb{Z}^3 \to \Lambda_2 \subset \mathbb{R}^2$



- Target Lattice Λ_t. Gram matrix of dual: $M_t = \begin{pmatrix} 1 & P \\ P & Q \end{pmatrix}$
- Let $\mathbf{a} = (1, a, b)$ $a < b$.
- Let $\mathbf{v_a} = (-a, 1, 0)/\|\mathbf{a}\|^2$, $\mathbf{v_b} = (-b, 0, 1)/\|\mathbf{a}\|^2$
- $\Lambda_{\mathbf{a}}^* = IntegerSpan\{\mathbf{v_a}, \ \mathbf{v_b}\}$
- Let $\mathbf{v}_{red} := \mathbf{v_b} - k\mathbf{v_a}$
- $k \approx b/a$, $-b + ka = \delta$
- $\mathbf{v}_{red} = (\delta, -k, 1)$ is shortest vector.

Construction: $\mathbb{Z}^3 \to \Lambda_2$ Continued

Approximation problem:

$$\frac{\langle \mathbf{v}_{red}, \mathbf{v}_{\mathbf{a}} \rangle}{1+a^2} = \frac{-a\delta - k}{1+a^2} \approx P$$

and

$$\frac{\langle \mathbf{v}_{red}, \mathbf{v}_{red} \rangle}{1+a^2} = \frac{\delta^2 + k^2 + 1}{1+a^2} \approx Q$$

With $k \sim a$, get two independent equations.
$-\delta/a \approx P$ and $\delta^2/a^2 + k^2/a^2 \approx Q$.

$$k/a \approx \sqrt{Q - P^2}$$

and

$$\delta/a \approx P$$

k, δ integer.

The Lifting Construction: Slide 1

Let $\mathbf{a} = (1, a_1, a_2, \ldots, a_n)$ and let $\Lambda_{\mathbf{a}} = \mathcal{P}_{\mathbf{a}}\mathbb{Z}^n$ Let $\Lambda^*_{\mathbf{a}}$ denote the dual lattice of $\Lambda_{\mathbf{a}}$.

1. $\Lambda^*_{\mathbf{a}} = \mathbb{Z}^n \bigcap \mathbf{a}^{\perp}$.
2. Generator matrix for $\Lambda^*_{\mathbf{a}}$

$$\mathbf{M}_{\mathbf{a}} := \begin{pmatrix} -a_1 & 1 & 0 & \ldots & 0 \\ -a_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \\ -a_{n-1} & 0 & \ldots & 0 & 1 \end{pmatrix} \qquad (1)$$

The Lifting Theorem: Statement

$\mathbf{M}_t = (m_{i,j})$: Rows are basis for Λ_t^*. Lower Triangular.
Let

$$\mathbf{M}_l = \begin{pmatrix} -[wm_{1,1}] & 1 & 0 & \dots & 0 \\ -[wm_{2,1}] & -[wm_{2,2}] & 1 & 0\dots & 0 \\ \vdots & \vdots & & \ddots & \\ -[wm_{n-1,1}] & -[wm_{n-1,2}] & \dots & -[wm_{n-1,n-1}] & 1 \end{pmatrix}.$$

Let

$$\tilde{\mathbf{M}}_l = \begin{pmatrix} -a_1 & 1 & 0 & \dots & & 0 \\ -a_2 & 0 & 1 & 0\dots & & 0 \\ \vdots & 0 & 0 & \ddots & \ddots & \vdots \\ -a_{n-2} & 0 & \dots & 0 & 1 & 0 \\ -a_{n-1} & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

be obtained by applying elementary row operations to $\mathbf{M}_l$. The Gram matrix of $\Lambda_{\mathbf{a}}^*$, with $\mathbf{a} = (1, a_1, a_2, \dots, a_{n-1})$, converges to that of Λ_t^*, as $w \to \infty$.

Lifting Theorem: Proof

Proof:

1. $\tilde{\mathbf{M}}_l$ shown in is a basis for $\Lambda^*_{\mathbf{a}}$.
2. $\tilde{\mathbf{M}}_l$ can be obtained from $\mathbf{M}_l$ by elementary row operations.
3. Let $\mathbf{G}_w := \mathbf{M}_l \mathbf{M}_l^{tr} / w^2$.
4. Then

$$\mathbf{G}_w = \mathbf{G}_t^* + \Gamma_w / w^2 \tag{2}$$

where the polynomial-in-w entries of the $n \times n$ matrix Γ_w are of highest degree 1. Thus $\Gamma_w / w^2 \to 0$ as $w \to \infty$.

Example: Target D_n

Applying the Lifting theorem to the following triangular basis for D_n^*

$$\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & \ddots & 0 & 0 \\ \vdots & & & 1 & 0 \\ 1/2 & 1/2 & \dots & \dots & 1/2 \end{pmatrix}; \tag{3}$$

we obtain

$$\mathbf{a}_n = \left(1, 2w, (2w)^2, \dots, (2w)^{n-1}, \frac{w((2w)^n - 1)}{(2w - 1)}\right).$$

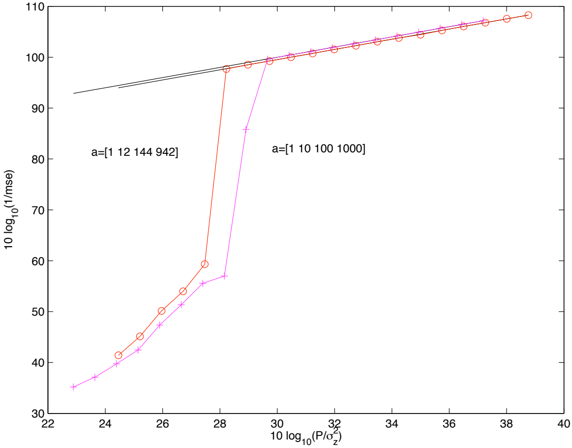
Target: E_8

$\mathbf{a} = (1, a_1, a_2, \ldots, a_8)$ is given by

$$\begin{aligned} a_1 &= 2w; \\ a_2 &= 2w^2 - w; \\ a_i &= w(a_{i-1} - a_{i-2}), \;\; i = 3, 4, \ldots, 7; \\ a_8 &= (w/2)\left(\sum_{i=1}^{7} a_i + 1\right) \end{aligned} \tag{4}$$

for w even.

Performance Gains



Summary

- $\texttt{FatStrut}(\mathbf{a})$ is the largest unobstructed cylinder whose axis contains 0 and $\mathbf{a}$.
- Objective. Find $\mathbf{a} \in \mathbb{Z}^n$ such that radius of fat strut is maximized.
- Initial motivation from communication theory, but problem is of independent interest.
- Upper and lower bounds on performance were derived.
- A general construction method, Lifting was described.